Chapter 4

Moments of a Random Variable

4.1. General definition: Let X be a random variable with PF $f(x)$ and $u(X)$ a function of X. We define the **Expected value** or **Expectation** of $u(X)$ as

$$E[u(X)] = \sum_{x=x_i} u(x)f(x) \tag{1}$$

if X is discrete,

$$E[u(X)] = \int_{-\infty}^{\infty} u(x)f(x)\,dx \tag{2}$$

if X is continuous, and

$$E[u(X)] = \int_A u(x)f(x)\,dx + \sum_{x=x_i} u(x)f(x). \tag{3}$$

if the distribution is mixed, where A is the set of points where X is continuous and x_i's the points where X is discrete.

In general, these expected values may or may not exist.

Example 4.1.1: Let the probability function of X be

$$f(-1) = 1/4; f(0) = 1/4;\ f(1) = 1/4,\ f(2) = 1/8;\ f(3) = 1/8.$$

Calculate $E(X)$ and $E[e^{tX}]$.

Solution:

$$\begin{aligned} E(X) &= (-1)(1/4) + (0)(1/4) + (1)(1/4) + (2)(1/8) + (3)(1/8) = 5/8 \\ E[e^{tX}] &= e^{-t}(1/4) + e^{0t}(1/4) + e^{t}(1/4) + e^{2t}(1/8) + e^{3t}(1/8) \\ &= (1/4)e^{-t} + (1/4) + (1/4)e^{t} + (1/8)e^{2t} + (1/8)e^{3t}. \end{aligned}$$

Example 4.1.2. Let the PDF of X be $f(x) = ce^{-cx}\ x > 0,\ c$ a positive constant. Calculate $E(X)$ and $E[e^{tX}]$.

Solution:

$$\begin{aligned} E(X) &= c\int_0^{\infty} xe^{-cx}\,dx = 1/c \\ E(e^{tX}) &= c\int_0^{\infty} e^{tx}e^{-cx}\,dx = c\int_0^{\infty} e^{-(c-t)x}\,dx = \frac{c}{c-t},\ t < c. \end{aligned}$$

For the first integral we used integration by parts. (Please verify the calculation.) The second integral does not exist if $t \geq c$.

Example 4.1.3: The CDF of X is

$$F(x) = \begin{cases} 0 & x < 0 \\ x/3 & 0 \leq x < 2 \\ 1 & x \geq 2. \end{cases}$$

Calculate $E(X)$ and $E(X^2)$.

Solution: There is a point mass at 2. $F(2-) = 2/3$ and $F(2+) = 1$. So $Pr(X = 2) = 1 - 2/3 = 1/3$. For $0 < x < 2$, $f(x) = F'(x) = 1/3$. Hence

$$\begin{aligned} E(X) &= \int_0^2 x(1/3)\,dx + (2)Pr(X = 2) = (1/6)(4) + (2)(1/3) = 4/3 \\ E(X^2) &= \int_0^2 x^2(1/3)\,dx + 2^2 Pr(X = 2) = (8/9) + (4)(1/3) = 20/9. \end{aligned}$$

The expected value of a constant function is that constant.

$$E(c) = c\int_{-\infty}^{\infty} f(x)dx = c(1) = c. \tag{4}$$

A similar argument holds for the discrete case (and the mixed) as well since the total of the probabilities has to be 1. $u(X) = c$ implies that whatever X be $u(X)$ equals a constant c and it cannot take any other value. Hence $Pr[u(X) = c] = 1$. That is, $u(X) = c$ with certainty and $E[u(X)] = c$.

4.2. Moments, Mean and Variance of random variable: The n-th moment of a random variable X is $E(X^n)$.

The first moment, $E(X)$, is called the **mean** or the **expected value** of X.

The n-th **central moment** of X is defined as $E[\{X - E(X)\}^n]$. The second central moment of X is called the **variance** of X and is denoted by $Var(X)$.

$$Var(X) = E[\{X - E(X)\}^2]. \tag{5}$$

Note that, since variance is the expected value of a square, it cannot be negative. The analogy between mass and probability is useful here. If you think of masses stretched out along the x-axis, the mean corresponds to the center of mass. The variance is a measure of how spread out from the center of mass are the masses.

One extreme is if the mass is concentrated at a single point, say, c. That is $Pr(X = c) = 1$. In that case, $E(X) = c$ and $Var(X) = E[(X - c)^2] = E(0) = 0$.[1]

The expected value of X has the units of X. That means, if X is duration, for example, in years, then $E(X)$ has the units of years. $Var(X) = E[\{X - E(X)\}]^2$, has the units of the square of X. Therefore $\sqrt{Var(X)}$ has the units of X. It is natural then to compare the square root of the variance to the mean to get an idea of the spread. We define the standard deviation (SD) and the coefficient of variation (CV) as

$$SD(X) = \sqrt{Var(X)} \tag{6}$$

$$CV(X) = \frac{SD(X)}{E(X)},\ E(X) \neq 0. \tag{7}$$

Example 4.2.1: The PF of X is $Pr(X = -c) = Pr(X = c) = 1/4,\ c > 2;\ Pr(X = 2) = 1/2;$. Calculate the coefficient of variation of X.

Solution:

$$\begin{aligned}
E(X) &= (-c)Pr(X = -c) + (c)Pr(X = c) + (2)Pr(X = 2) = 1 \\
Var(X) &= E[\{X - E(X)\}^2] = (c+1)^2(1/4) + (c-1)^2)(1/4) + (2-1)^2(1/2) = \frac{c^2+2}{2} \\
SD(X) &= = \sqrt{Var(X)} = \sqrt{\frac{c^2+2}{2}} \\
CV(X) &= \frac{SD(X)}{E(X)} = \sqrt{\frac{c^2+2}{2}}.
\end{aligned}$$

This shows that as the masses get farther and farther from each other (c becomes large), even though the mean stays the same, the spread gets larger.

[1] Conversely, suppose that $Var(X) = 0$. This means $E[\{X - E(X)\}^2] = 0$. $\{X - E(x)\}^2$, being a square, cannot be negative. It has to positive or zero. If it is positive with a positive probability, $E[\{X - E(X)\}^2]$ will have to be positive. Therefore $E[\{X - E(X)\}^2] = 0$ implies that $Pr[\{X - E(X)\}^2 > 0] = 0$. Hence $Pr[\{X - E(X)\}^2 = 0] = 1$. That means $X = E(X)$ with certainty implying that, if $E(X) = c$, then $Pr(X = c) = 1$. Thus $Var(X) = 0$ if and only if X takes a value c with certainty.

Linearity of the expected value: If a and b are constants, then

$$E[au(X) + bv(X)] = aE[u(X)] + bE[v(X)]. \tag{8}$$

In particular,

$$E(aX + b) = aE(X) + b. \tag{9}$$

Let us prove this for the continuous case. For the discrete case the same proof works with sums replacing integrals.

$$\begin{aligned} E[au(X) + bv(X)] &= \int_{-\infty}^{\infty} [au(x) + bv(x)]f(x)\,dx \\ &= a\int_{-\infty}^{\infty} u(x)f(x)\,dx + b\int_{-\infty}^{\infty} v(x)f(x)\,dx = aE[u(X)] + bE[v(x)]. \end{aligned}$$

An alternative expression for variance: From Eq.(5) and Eq.(9) we can get the following useful expression for the variance.

$$\begin{aligned} Var(X) &= E[\{X - E(X)\}^2] = E[X^2 - 2XE(X) + \{E(X)\}^2] \\ &= E(X^2) - 2E(X)E(X) + \{E(X)\}^2 \\ &= E(X^2) - \{E(X)\}^2. \end{aligned} \tag{10}$$

Variance is <u>NOT</u> linear. In fact, for constants a and b,

$$\begin{aligned} &Var(aX + b) = E[(aX + b)^2] - \{E(aX + b)\}^2 \\ &= E[a^2X^2 + 2abX + b^2] - \{aE(X) + b\}^2 \\ &= a^2E(X^2) + 2abE(X) + b^2 - a^2\{E(X)\}^2 - 2abE(X) - b^2 \\ &= a^2[E(X^2)] - a^2\{E(X)\}^2 = a^2Var(X). \end{aligned} \tag{11}$$

The standard notation is μ for the mean, $\sigma^2,\ \sigma \geq 0$ for the variance and σ for the standard deviation.

The following example shows an application of Eqs.(9) and (11).

Example 4.2.1: This year a company will have fixed expenses of 100. The rest of the expenses is a random variable with PDF $f(x) = (1/300), 0 < x < 300$. Next year the fixed expenses will increase by 20% whereas the rest of the expenses will increase by 10%. Calculate the following:

1. The expected value of the total expenses this year,

2. The variance of the total expenses this year,
3. The expected value of the total expenses next year,
4. The variance of the total expenses next year.

Solution: Let X be the rest of the expenses this year. Then the total expenses this year is $100 + X$ and the total expenses next year is $120 + 1.1X$.

$$E(X) = \int_0^{300} x(1/300)\,dx = (1/600)(300)^2 = 150$$
$$E(X^2) = \int_0^{300} x^2(1/300)\,dx = (1/900)(300)^3 = 30,000$$
$$Var(X) = 30,000 - 150^2 = 7,500$$

1. This year: $E(100 + X) = 100 + E(X) = 250$
2. This year: $Var(100 + X) = Var(X) = 7,500$
3. Next year: $E(120 + 1.1X) = 120 + 1.1E(X) = 120 + (1.1)(150) = 285$
4. Next year: $Var(120 + 1.1X) = (1.1)^2 Var(X) = (1.21)(7,500) = 9,075.$

4.3. The Moment Generating Function: This is a device by which you can generate the moments of a random variable. The **Moment Generating Function**, or MGF for short, of a random variable X is defined by

$$M(t) = E[e^{Xt}] \tag{12}$$

You should always keep this definition in mind.

As we mentioned for the PDF and CDF, if we want to emphasize that this is the MGF of X, we will write $M_X(t)$.

In general the MGF may or may not exist. If it exists, we can then formally write, using the power series for the exponential function and the linearity of the expectation,

$$M(t) = E\left\{\sum_0^\infty \frac{t^n}{n!} X^n\right\} = \sum_0^\infty E(X^n)\,\frac{t^n}{n!}.$$

The right-hand side is the Taylor series around the origin for the function $M(t)$. Recall that the Taylor series around the origin can be written as

$$M(t) = \sum_0^\infty \left\{\frac{d^n}{dt^n} M(t)|_{t=0}\right\} \frac{t^n}{n!}.$$

Comparing the above two equations, we conclude that

$$E(X^n) = \frac{d^n}{dt^n} M(t)|_{t=0}. \tag{13}$$

In particular

$$\begin{aligned} M(0) &= 1 & (14)\\ M'(0) &= E(X) & (15)\\ M''(0) - [M'(0)]^2 &= Var(X). & (16) \end{aligned}$$

Since the coefficients of the Taylor series for $M(t)$ are the moments of X, $M(t)$ "generates" the moments of X. Hence its name.

We have already derived moment generating functions in Examples 4.1.1. and 4.1.2.

If X is a discrete random variable with PF, $Pr(X = x_i = p_i,\ i = 1, 2, \cdots,$ then

$$M_X(t) = e^{tx_1} p_1 + e^{tx_2} p_2 + \cdots. \tag{17}$$

In the discrete case, if you are given the MGF in the form of the above equation, you can read on sight the PF of X. $Pr(X = x_i$ is the coefficient of e^{tx_i}.

Example 4.3.1: Suppose the MGF of X is

$$0.1e^{-t} + 0.2 + 0.3e^{5t} + 0.4e^{8t}.$$

You can immediately conclude that the coefficient of e^{-1t}, which is 0.1 is $Pr(X = -1)$, the coefficient of e^{0t}, which is 0.2 is $Pr(X = 0)$, and so on.

Hence the PF is

$$Pr(X = -1) = 0.1;\ Pr(X = 0) = 0.2;\ Pr(X = 5) = 0.3;\ Pr(X = 8) = 0.4.$$

In the continuous case

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx.$$

Example 4.3.2: Find the MGFs of the following random variables:

1. $Pr(X = 0) = q;\ Pr(X = 1) = p = 1 - q$
2. $Pr(X = n) = p\,q^n\ n = 0, 1, 2, \cdots,\ p = 1 - q$
3. $f(x) = \frac{1}{b-a}$ for $a < x < b$ and 0 otherwise.
4. $f(x) = ce^{-cx},\ x > 0.$

Solution:

(1). $M(t) = E(e^{tX}) = (e^{0t})Pr(X = 0) + e^{1t}Pr(X = 1) = q + pe^t$

(2). $M(t) = E(e^{tX}) = p\sum_0^\infty e^{nt}q^n = p\sum_0^\infty (qe^t)^n = \dfrac{p}{1 - qe^t};\ qe^t < 1$ (geometric series)

(3). $M(t) = \dfrac{1}{b-a}\displaystyle\int_a^b e^{tx}\,dx = \dfrac{e^{bt} - e^{at}}{(b-a)t}$

(4). (See Example 4.1.2.) $M(t) = c\displaystyle\int_0^\infty e^{-cx}e^{tx}\,dx = \dfrac{c}{c-t},\ t < c.$

Example 4.3.2: The MGF of a random variable X is

$$M(t) = (1/2)e^t + \frac{1}{2-t}.$$

Calculate $Var(X)$.

Solution: Use Eqs.(14)-(16).

$$\begin{aligned} M'(t) &= (1/2)e^t + \frac{1}{(2-t)^2};\ M'(0) = 1/2 + 1/4 = 3/4 \\ M''(t) &= (1/2)e^t + \frac{2}{(2-t)^3};\ M''(0) = (1/2) + (1/4) = 3/4 \\ Var(X) &= (3/4) - (3/4)^2 = 3/16. \end{aligned}$$

4.4: A couple of useful results: Suppose that X is a random variable that **assumes only nonnegative values.** That means the PDF $f(x)$ is 0 for $x < 0$. Let us think of X as a loss. Suppose an insurance will pay the entire loss up to a maximum of m. If we denote the payment by Y, then

$$Y = \begin{cases} X & X \le m \\ m & X > m. \end{cases}$$

In other words $Y = \min(X, m)$, meaning Y is the smaller of X and m. One sometimes denotes $\min(X, m)$ by $X \wedge m$. Now

$$E(Y) = E(X \wedge m) = \int_0^m x f(x)\,dx + m \int_m^\infty f(x)dx.$$

We can derive a very useful expression (one that will be useful in later exams as well) for $E(Y)$ using integration by parts. We set $f(x) = u'$. That means we need an antiderivative for $f(x)$. It is true that $F(x)$ is one antiderivative, but that goes to 1 at infinity. On the other hand, if we choose $-1 + F(x)$ as the antiderivative of $f(x)$, that goes to zero at infinity. Thus

$$\begin{aligned}
E(Y) &= \int_0^m x[-1 + F(x)]'\,dx + mPr(X > m) \\
&= x[-1 + F(x)]|_0^m - \int_0^m x'[-1 + F(x)]\,dx + m[1 - F(m)] \\
&= m[-1 + F(m)] - \int_0^m [-1 + F(x)]dx + m[1 - F(m)] \\
&= \int_0^m [1 - F(x)]dx \qquad (18)
\end{aligned}$$

Example 4.4.1: The pdf of a loss is $f(x) = 0.5e^{-0.5x},\ x > 0$. An insurance will pay the entire loss up to a maximum of 1. Calculate the expected value of the payment.

Solution: It is a good idea to keep in mind that if $f(x) = ce^{-cx}, x > 0$ then the CDF is $F(x) = 1 - e^{-cx}$. Because

$$F(x) = \int_0^x ce^{-cy}\,dy = -e^{-cy}|_0^x = 1 - e^{-cx}.$$

So for this example, $F(x) = 1 - e^{-0.5x}$.

$$E(X \wedge 1) = \int_0^1 [1 - F(x)]\,dx = \int_0^1 e^{-0.5x}\,dx = \frac{1 - e^{-0.5}}{0.5}.$$

Suppose that we further assume that $\lim_{m\to\infty} m[1 - F(m)] = 0$ and let m go to infinity in Eq.(18). Since $X \wedge \infty$, which is the smaller of X and infinity, is X, we get the following formula for the mean of X.

$$E(X) = \int_0^\infty [1 - F(x)]\,dx. \qquad (19)$$

What is interesting is that this result holds also for discrete and mixed distributions.

Example 4.4.2: Consider Example 4.1.3. The CDF given there is 0 for $X < 0$. So we can use Eqs.(19) to get

$$\begin{aligned} 1 - F(x) &= \begin{cases} 1 - x/3 & 0 < x < 2 \\ 0 & x > 2 \end{cases} \\ E(X) &= \int_0^2 \left\{1 - \frac{x}{3}\right\} dx = 2 - (1/6)(4) = 4/3 \end{aligned}$$

Remarks

- The variance is always non-negative.

$$Var(X) = E(X^2) - [E(X)]^2 = E[\{X - E(X)\}^2]. \tag{20}$$

- If a and b are constants, then

$$Var(aX + b) = a^2 Var(X). \tag{21}$$

- If X assumes only non-negative values, then

$$E(X) = \int_0^\infty [1 - F(x)]\, dx. \tag{22}$$

  and

$$E[\min(X, a)] = \int_0^a [1 - F(x)]\, dx. \tag{23}$$

- If $M(t)$ is the Moment Generating Function of X, then

$$\begin{aligned} M(0) &= 1 && (24) \\ M'(0) &= E(X) = \frac{d}{dt} \ln M(t)|_{t=0} && (25) \\ M''(0) &= E(X^2) && (26) \\ \frac{d^2}{dt^2} \ln M(t)|_{t=0} &= Var(X). && (27) \end{aligned}$$

Problems

1. A random variable, X, has the PF, $Pr(X = k) = 0.1,\ k = 0, 1, 2, \cdots 9$. Calculate $Var(X)$. (4.5, 8.25)

   The following information is for Problems 2-5:

   The probability function of a loss X is given by:

x	0	100	150	200	300	400	600
$Pr(X = x)$	0.3	0.2	0.2	0.1	0.1	0.05	0.05

2. Calculate
   (a) $Pr(120 < X < 400)$ (0.4)
   (b) $E(X)$ (150)
   (c) The standard deviation of X (151.7)

3. An insurance will pay the excess if any of the loss over 160. Calculate the expected payment. (52)

4. Given that the payment is positive, find the conditional probability function of the payment.

5. Given that the payment is positive, calculate the conditional expected value of the payment. (173.3)

6. The PDF of a random variable, X, is

$$f(x) = c|x - 2|,\ 0 < x < 4, \quad \text{and 0 elsewhere.}$$

   Calculate $E(X)$ and $Var(X)$. (2;2)

7. Let $E(X^n) = p_n$. Show that

$$E[(X - p_1)^3] = p_3 - 3p_2\, p_1 + 2\, p_1^3.$$

   This entity is called the third central moment of X, denoted by $TCM(X)$.

8. Show that

$$E(X) = \frac{d}{dt}\ln M(t)|_{t=0}, \quad Var(X) = \frac{d^2}{dt^2}\ln M(t)|_{t=0} \text{ and } TCM(X) = \frac{d^3}{dt^3}\ln M(t)|_{t=0}.$$

(See Problem 4 of Chapter 0.)

9. For each of the random variables in Example 4.3.2, calculate the mean and the variance.

10. A loss X this year has the PDF $f(x) = 2(1+x)^{-3}, x > 0$. Next year the loss will increase by 20%. Next year a deductible of 1 will be imposed. That means a payment of the amount of loss minus 1 will be made if the amount of loss exceeds 1 and nothing will be paid if it is less than or equal to 1. Calculate the expected payment next year. (0.6545)

11. The cost of running a business has two components. This year there is a fixed cost of 15 and a random cost, X. Next year the fixed cost will be 19 and the random cost will increase by 20%.

    Calculate the ratio of the variance of the total cost next year to the variance of the total cost this year. (1.44)

12. *An insurance company's monthly claims are modeled by a continuous, positive random variable X, whose probability density function is proportional to $(1+x)^{-4}$, where $0 < x < \infty$. Determine the company's expected monthly claims. (1/2)

13. *A probability distribution of the claim sizes for an auto insurance policy is given in the table below:

Claim size	Probability
20	0.15
30	0.10
40	0.05
50	0.20
60	0.10
70	0.10
80	0.30

What percentage of the claims are within one standard deviation of the mean claim size? (45%)

14. *An insurance policy reimburses a loss up to a benefit limit of 10. The policyholder's loss, Y, follows a distribution with density function $f(y) = 2/y^3$ if $y > 1$ and 0 otherwise. What is the expected value of the benefit paid under the insurance policy? (1.9)

15. An insurance policy reimburses the excess of a loss over 2. The policyholder's loss, Y, follows a distribution with density function $f(y) = 2/y^3$ if $y > 1$ and 0 otherwise. What is the expected value of the benefit paid under the insurance policy? (1/2)

16. *A recent study indicates that the annual cost of maintaining and repairing a car in a town in Ontario averages 200 with a variance of 260 . If a tax of 20% is introduced on all items associated with the maintenance and repair of cars (i.e., everything is made 20% more expensive), what will be the variance of the annual cost of maintaining and repairing a car? (374.4)

17. *An actuary determines that the claim size for a certain class of accidents is a random variable, X, with moment generating function $M_X(t) = (1-2,500\,t)^{-4}$. Determine the standard deviation of the claim size for this class of accidents. (5,000)

18. *A manufacturer's annual losses follow a distribution with density function $f(x) = (2.5)(0.6)^{2.5}x^{-3.5},\ x > 0.6$ and 0 otherwise.

    To cover its losses, the manufacturer purchases an insurance policy with an annual deductible of 2. What is the mean of the manufacturer's annual losses not paid by the insurance policy? (0.934)

19. *The warranty on a machine specifies that it will be replaced at failure or age 4, whichever occurs first. The machine's age at failure, X, has density function $f(x) = 1/5,\ 0 < x < 5$ and 0 otherwise. Let Y be the age of the machine at the time of replacement. Determine the variance of Y. (1.71)

20. *An insurance policy on an electrical device pays a benefit of 4000 if the device fails during the first year. The amount of the benefit decreases by 1000 each successive year until it reaches 0. If the device has not failed by the beginning of any given year, the probability of

failure during that year is 0.4. What is the expected benefit under this policy? (2,694)

21. *Let X be a continuous random variable with density function $f(x) = 0.1\,|x|$ for $-2 < x < 4$ and 0 otherwise. Calculate the expected value of X. (1.87)

22. A random variable X has the cumulative distribution function:

$$F(x) = \begin{cases} 0 & x < 1 \\ (1/2)(x^2 - 2x + 2) & 1 \leq x < 2 \\ 1 & x > 2. \end{cases}$$

Calculate the variance of X. (5/36)

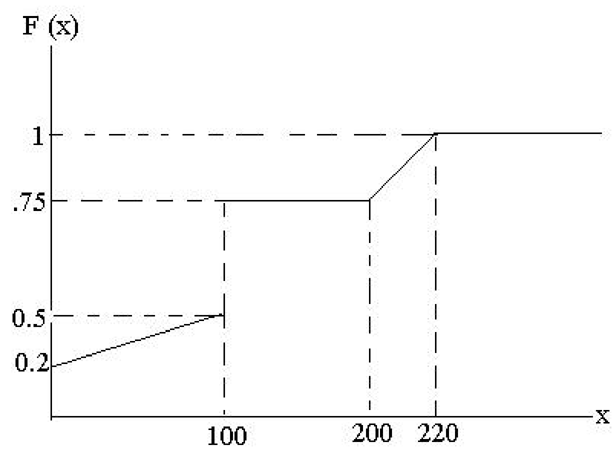
23. An insurance will pay the entire loss up to a maximum of 20. The CDF of the loss is:

$$F(x) = 1 - \left(\frac{10}{10+x}\right)^3.$$

Calculate the expected value of the payment. (40/9)

The following information is for Problems 24 and 25.

A random loss X which assumes only non-negative values has the CDF shown in the graph below. All the lines are straight lines.



24. Calculate the expected value of the loss. (92.5)

25. The insurance will pay the entire loss up to a maximum of 210. Calculate the expected value of the payment. (91.875)

26. The n-th moment of a random variable, X, about the point a is defined as $E[(X-a)^n]$. You are given:

    - The second moment of X about the point a is 3.
    - The second moment of X about the point $-a$ is 11.
    - The second moment of X about the point $2a$ is 2.

    Calculate the variance of X. (2)

27. The probability that a loss will occur is 0.1. If the loss occurs, the amount of the loss has the density function,

$$f(x) = cx^{-5}, \ x > 1$$

    and 0 otherwise. An insurance will pay the entire amount of loss.

    Calculate the variance of the payment. (0.1822)

28. Fifty-two percent of a population are females and the rest males. The number of automobile insurance claims, N, for females in a given year has the probability function

$$Pr(N = n) = (0.4)(0.6)^n, \ n = 0, 1, 2, \cdots.$$

    The number of automobile insurance claims, M, for males in a given year has the probability function

$$Pr(M = m) = 2^m 3^{-(m+1)}, m = 0, 1, 2, \cdots.$$

    Let X be the number of claims in a year made by an individual chosen at random from this population.

    (a) Determine the Probability Function of X.
    (b) Determine the MGF of X.
    (c) Calculate $E(X)$. (1.74)