Session 12

Reading assignment: Chapter II-1 and SOA Study Note.

12.1. Review of properties of the exponential distribution:

- The pdf is $f(t) = \lambda e^{-\lambda t}, t > 0$
- $Pr(T > t) = e^{-\lambda t}$
- $E(T) = 1/\lambda,\ Var(T) = 1/\lambda^2$
- The exponential random variable is memoryless. That is,

$$Pr(T > t + s | T > t) = e^{-\lambda s}$$

- The sum of n independent, identical exponential variables has a gamma distribution.

Exercise for reviewing the exponential distribution: True or false? If X has an exponential distribution, then $E(X^2|X > 1) = E[(X + 1)^2]$.

12.2: The Poisson Process: This write up covers all the material in the Study Note. But I think itis nicer to consider the homogeneous process first and then the variations.

Main ideas :

- If something is moving and you can say precisely where it is at any time, then there is little that is interesting. On the other hand, if you can tell only the probability distribution of where it is at any given time, then it is interesting and you have a stochastic process.
- If the thing that is moving can only be in a countable number of places (states), then there is some simplification. We can then label the states by integers. Furthermore, if given that the process is in state i at a certain time s, the conditional probability that it will go to state j later does not depend upon history (i.e., where it was before s), then the process is called a Markov process.

- Suppose we can number the states as $0, 1, 2, \cdots$ If, after leaving state n, the process can go only to state $n+1$, then the process is called a counting process; because, you can think of the process as a counter which counts the number of occurrences.

- We can think of the counting process as follows: It is in state n. It stays there for some time, T_n, which is a random variable. At the end of time T_n it goes to state $n+1$. For example, you sit at the entrance to a subway station and count the number of people that come in. If 20 people have come in, then the process is in state 20. You wait. The time, T_{20}, you wait until the next person comes in is a random variable. At the end of that time, the process goes to state 21. If the T_n's are all independent and identically distributed exponentially with mean $1/\lambda$, then the process is called a **Poisson Process with rate** λ.

- For a Poisson process the waiting time until the n-th event has a gamma distribution. It is the sum of n identically distributed independent exponential variables, each with mean $1/\lambda$.

- Always identify the Poisson parameter. The rate λ is per unit time. The expected number of occurrences in a period t is λt and the variance is λt.

- In a Poisson process the number of events in two disjoint time intervals are independent (caled the "independent increments" property).

- In a Poisson process the distribution of the number of events in a time interval depends only on the length of the interval and not on its location. The distribution of the number of events in the interval $(a, a+t)$ is Poisson with mean λt.

- The sum of independent Poisson is Poisson.

- Decomposition: Let us give a concrete example. Consider a compound Poisson process with rate λ and discrete claim amount distribution, $Pr(X = x_i) = p_i, \ i = 1, 2, \cdots, m$. Then each claim can be thought of as of type i with probability p_i. Then the number of claims of size i is also Poisson with rate λp_i.

- Non-homogeneous Poisson process: Here the rate $\lambda(t)$ is a non-constant function of t. Since $\lambda(t)$ is a rate, the expected number of events over an interval is the integral of $\lambda(t)$ over that interval.

The following example deals with all the ideas you need to know.

Example: Suppose that customers arrive at a Poisson rate of 120 per hour.

1. Calculate the probability that there will be at least two customers in the next two minutes.

2. Given there are five customers in the next two minutes, calculate the expected number of customers in the next five minutes.

3. If on the average a third of the customers are male (independently of the total number of customers), given that three males arrived in three minutes calculate the expected number of customers in those three minutes.

4. If the Poisson rate, instead of being a constant, is 120 from 10 A.M. until 11 A.M. and then it linearly reduces to 100 in the next half hour, what is the distribution of number of customers from 10:30 A.M. till 11:30 A.M?

Solution:

1. The expected number per two minutes is 4. So the probability that there will be at least two customers in the next two minutes is $1 - Pr(N \leq 1) = 1 - e^{-4}(1 + 4)$.

2. The number in the next two minutes and the number in the three minutes after that are independent. The expected number in three minutes is 6. So the expected number in five minutes given that there are 5 in the first two minutes is $6 + 5 = 11$.

3. The number of females has a Poisson distribution with mean 80 per hour or 4 per three minutes. Since three males have arrived, the total expected number is 7.

4. This is non-homogeneous Poisson with mean equal to the integral of the function $\lambda(t)$ over the interval from 10:30 A.M. to 11:30 A.M. That is just the area under the curve, which is $(1/2)(120) + (1/2)(1/2)(120 + 100) = 115$.

Basics Quiz

1. The number of events is a Poisson process with rate 2. Find the probability that there will be exactly two events between 3 units and 3.5 units of time.

   (A) 0.18 **(B)** 0.19 **(C)** 0.20 **(D)** 0.21 **(E)** 0.22

2. For a Poisson process, $\{N(t), t \geq 0\}$ with rate 2, calculate $Pr[N(3) - N(2) < 2|N(1) = 2)]$.

   (A) 0.27 **(B)** 0.41 **(C)** 0.55 **(D)** 0.68 **(E)** 0.82

3. In a compound Poisson aggregate claims process the number of claims has rate 3. The severity distribution is exponential with mean 2. If $S(t)$ is aggeregate claims up to time t. calculate $Var[S(4) - S(2)]$.

   (A) 12 **(B)** 24 **(C)** 36 **(D)** 48 **(E)** 60

4. Trains arrive at a station at a Poisson rate of 12 per hour. Given that 5 trains have arrived between 9 A.M. and 9:20 A.M., calculate the expected number of trains that arrive between 9 A.M. and 9.40 A.M.

   (A) 3 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 9

5. Trains arrive at a station at a Poisson rate of 12 per hour. 25% of these trains are express trains. 75% are local, independently of the arrival of of express trains. A local train has just left. Calculate the expected time in minutes until the arrival of an express train.

   (A) 5 **(B)** 10 **(C)** 15 **(D)** 20 **(E)** 25

Quiz 12

1. Electronic mail messages arrive at a Poisson rate of one per minute. Find the probability that between 10:00 A.M. and 10:02 A.M. there are at least two messages.

   (A) 0.57 **(B)** 0.59 **(C)** 0.61 **(D)** 0.63 **(E)** 0.65

   Use the following information for Problems 2 and 3.

   John and Mary have independent businesses but share an office. Telephone calls arrive in the office at a Poisson rate of five per hour. On the average the number of calls for John are fifty percent more than those for Mary. (Every call is specifically for John or for Mary and not for both). Mary takes a coffee break between 10:00 A.M. and 10:15 A.M.

2. Find the probability that there were no calls for John while Mary had her coffee break.

   (A) 0.47 **(B)** 0.49 **(C)** 0.51 **(D)** 0.53 **(E)** 0.55

3. Find the probability that there are (at least) two calls for John before there are three calls for Mary.

   (A) 0.78 **(B)** 0.80 **(C)** 0.82 **(D)** 0.84 **(E)** 0.86

4. The number of claims that are filed with an insurance company follows a Poisson process. The mean time between successive claims is 3 days. Find the probability that the second claim will arrive on the fourth day. (Do this two ways. First note that the time of arrival, S_2, of the second claim has a gamma distribution. Calculate $Pr(3 < S_2 < 4)$. Alternatively, the second claim arrives on the fourth day if and only if there are no claims in the first three days and there are at least two claims on the fourth day or if there is one claim in the first three days and there are one or more claims on the fourth day. Now use the fact that the number of claims in the first three days and the number of claims in the fourth day are independent.)

   A) 0.12 **(B)** 0.14 **(C)** 0.16 **(D)** 0.18 **(E)** 0.20

5. The number of traffic accidents along a stretch of a highway is a Poisson process. On week days, on the average, during the rush hours, between 7 A.M. and 9 A.M. and between 3 P.M. and 5 P.M. the rate of accidents is 1 per two days. At other times the rate of accidents is 1 per five days. On a given week day find the probability that there is at least one accident.

   (A) 0.14 **(B)** 0.16 **(C)** 0.18 **(D)** 0.20 **(E)** 0.22

6. Pieces of mail ready to be picked up are placed in a box at a Poisson rate of 20 per hour. A messenger arrives and picks up whatever mail is in the box. The number of hours between successive arrivals of a messenger is uniformly distributed over $(0, 0.5)$. A messenger has just picked up the mail. Calculate the variance of the number of pieces of mail that will be picked up by the next messenger.

   (A) 10.3 **(B)** 11.3 **(C)** 12.3 **(D)** 13.3 **(E)** 14.3

7. Lucky Tom lost his job as a walker for the SOA and is panhandling in front of Union Station. People pass him by at a Poisson rate of 15 per minute. 60% of the people give him nothing, 20% give him some change but no more than a quarter and the rest give him more than a quarter. Calculate the probability that over a period of 20 seconds somebody will give Lucky Tom some money.

   (A) 0.78 **(B)** 0.80 **(C)** 0.82 **(D)** 0.84 **(E)** 0.86

8. In a non-homogeneous Poisson process with rate function

$$\lambda(t) = \begin{cases} \frac{1}{50-t} & 0 \le t < 50 \\ 0 & t \ge 50. \end{cases}$$

   calculate the expected time until the occurrence of the first event.

   (A) 23.5 **(B)** 24.0 **(C)** 24.5 **(D)** 25.0 **(E)** 25.5

Solutions to Basics Quiz

1. The number of events in one half unit of time is Poisson with means $(1/2)(2) = 1$. Therefore the desired probability is

$$(1/2)e^{-1} = 0.1839.$$

Answer: A

2. Since the process is Poisson, the number events in the disjoint intervals $(0, 1)$ and $(2, 3)$ are independent. Therefore

$$Pr[N(3)-N(2) < 2|N(1) = 2)] = Pr[N(3)-N(2) < 2] = e^{-2}+2e^{-2} = 0.406.$$

Answer: B

3. The number of claims between times 2 and 4 is Poisson with mean 6. The second moment of the severity is $(2)2^2 = 8$. Therefore

$$Var[S(4) - S(2)] = (6)(8) = 48.$$

Answer: D

4. The number that arrive between 9:20 and 9:40 is independent of the number that arrive between 9 and 9:20. The expected number that arrive between 9:20 and 9:40 is $(1/3)(12) = 4$. Therefore the expected number between 9 and 9:40 is 9.

Answer: E

5. The number of express trains is Poisson at rate 3 per hour, independently of the number of locals. Therefore the expected time of arrival of the next express is 20 minutes.

Answer: D