Session 4 - Brownian Motion

Reading Assignment: Sections 4.1 - 4.5. (Correspondence with McDonald Chapter 20, up to Page 665.)

4.0. Some basic stuff:

- The normal distribution:
  - The symbol, $X \sim \mathcal{N}(a, b)$ stands for the statement, X has a normal distribution with mean a and **variance** b. The CDF of the standard normal PDF is denoted by $N(z)$. Another symbol is $\Phi(z)$. We will use the first because McDonald uses it.

    $$N(z) \equiv \Phi(z) = \frac{1}{2\pi} \int_{-\infty}^{z} e^{-\frac{x^2}{2}}\, dx.$$

    You will be provided with a normal CDF table in the exam. Please download it from the SOA and practice with it. That table is rather coarse. According to the SOA, you are supposed to choose the nearest z-value in the table and read the probability. For instance to find $Pr(Z \leq 0.759)$ read the value from the table for $Pr(Z \leq 0.76)$. This can lead to round off errors.

    Secondly for negative arguments you have to use the fact that

    $$N(-a) = 1 - N(a).$$

  - If $X \sim \mathrm{N}(\mu, \sigma^2)$ then

    $$Pr(X \leq x) = N\left(\frac{x - \mu}{\sigma}\right).$$

  - In one place in the Study Guide I use the moment generating function of a normal variable. This makes easy the understanding of processes followed by powers of $S(t)$ if $S(t)$ follows geometric Brownian motion. The MGF is given by

    $$M_X(t) = \exp\{\mu t + \frac{\sigma^2}{2} t^2\}.$$

- It is an important fact that if X_1 and X_2 are independent normal variables with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2 respectively, then $X_1 + X_2$ is normal with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. This is readily shown by the use of the MGF.

- **Smallness and asymptotics:** Perhaps the most fundamental idea in Itô's lemma is that you keep terms of the order of dt and neglect anything smaller. This means anything that is of the order of a power of dt that is higher than 1 is neglected. There is a subtle distinction between Δt and dt. The former is to be thought of as a small entity and the latter as an infinitesimal. If you think both of them as small that is fine too. I suggest not worrying about these details but having a grasp of "smallness." The reason this is extremely important is because the whole theory is based on the assumption that for small increments Δx and small times Δt, $\Delta x \sim \sqrt{\Delta t}$.

- **Risk-neutrality, no-arbitrage etc.:** They are related. Crudely speaking, prices are determined by saying that the expected return from your investment will be the same as if you had put the money in a risk-free investment, such as a zero-coupon bond. As I have pointed out in the last two sessions, the no-arbitrage analysis is equivalent to manipulating expressions so as to get rid of risk.

- **About return on an investment:** Suppose you deposit $S(0)$ dollars in an account that gives you $r\%$ annual interest. Then in one year you will have $S(1) = S(0)e^r$ dollars[1]. $r = \ln[S(1)/S(0)]$ is the return from the investment. If you invested that $S(0)$ in a stock then its value in one year, $S(1)$, is a random variable. Still, $\ln[S(1)/S(0)]$ is the return from the investment, but the return now is a random variable. The risk-neutral principle will tell you that the expected return should be $r - \delta$.

4.1. Quick lesson on Brownian Motion:

- Brownian motion is a limit of a random walk.

- Random walk is often described as follows (This picture is mainly of humorous value and not related to reality !): A drunk staggers out a

[1] r is the same as the force of interest.

public house and decides to walk home. He thinks he is going home but in reality it is equally likely that he takes a step towards home as a step away from home. Imagine that all the steps are along the same straight line, each step is of distance y and takes exactly the same time, h, and that the steps are all independent. As t and h approach 0 in such a way that $h/\sqrt{t}$ is constant, σ, then you get Brownian motion. The poor guy will have to take those steps extremely fast!

- We denote this limiting process by $\{X(t), t \geq 0.\}$ If $\sigma = 1$, we call the process a standard Brownian motion and denote it by $\{Z(t), t \geq 0\}$. $Z(t) \sim \mathcal{N}(0, t)$. and we use differentials to denote the Brownian motion $\{X(t), t \geq 0.\}$ by
$$dX(t) = \sigma dZ(t). \tag{1}$$
and
$$E(dX) = 0;\ Var(dX) = \sigma^2 dt. \tag{2}$$

- From the way we set this up, there are two important properties of Brownian motion, namely, the independent increments property and the stationary increments property. Increment is the change in $X(t)$ over an interval of time. Independent increments means the increments over disjoint intervals are independent. For example, $X(3) - X(0)$ and $X(6) - X(3)$ are independent. Stationary increments means the increment over an interval of time depends only on the length of the interval and not on the position of the interval. Thus $X(7) - X(4)$ has the same distribution as $X(3) - X(0)$. These properties should be effectively used in solving problems. Example 4.1.1 in the Study Guide has the standard type of problems using them.

Example 4.1.1: $\{Z(t), t \geq 0\}$ is standard Brownian motion. Calculate $E[Z(2)Z(3)]$.

Solution: Use the independence of $Z(2) = Z(2) - Z(0)$ and $Z(3) - Z(2)$.

$$\begin{aligned} E[Z(2)Z(3)] &= E[Z(2)\{Z(2) + Z(3) - Z(2)\}] = E[Z(2)^2] + E[Z(2)\{Z(3) - Z(2)\}] \\ &= E[Z(2)^2] + E[Z(2)]E[Z(3) - Z(2)] \quad \text{(Independence)} \\ &= Var[Z(2)] = 2, \end{aligned}$$

since$E[Z(2)] = 0$ and $Var[Z(2)] = 2$.

- Orders of magnitude. We neglect all terms that are smaller than dt but keep terms of the order of dt. For example, $E[(dX)^2] = Var(dX) + E^2(dX) = \sigma^2 dt.(E(dx) = 0.)$ This term is kept.

  Whereas $E[(dX)^2] = \sigma^2 dt$, $Var[(dX)^2]$ is of the order of dt^2 and so we neglect the latter. This tells us that

  $$(dX)^2 = \sigma^2 dt, \tag{3}$$

  is essentially not a random variable but a real number. This principle of keeping all terms of the order of dt and nothing smaller is crucial to deriving Itô's lemma.

4.2. Arithmetic Brownian motion: This process is sometimes called Brownian motion with drift. You can add a "drift" to the Brownian motion. Crudely speaking, you may think of Brownian motion as a process making rapid movements about the origin. (The expected position is the origin.) You can generalize this by considering a process that makes rapid movements about a fixed linear trajectory. The drift rate is the deterministic velocity. You then have

$$X(t) = \alpha t + \sigma Z(t). \tag{4}$$

or in infinitesimal form

$$dX(t) = \alpha dt + \sigma dZ(t). \tag{5}$$

α is called the drift coefficient or drift rate. Then $[X(t) - \alpha t]/\sigma$ has a normal distribution with mean 0 and variance t, or$[X(t) - \alpha t]/\sigma\sqrt{t}$ is standard normal

Example 4.2.1: $\{X(t), t \geq 0\}$ is an arithmetic Brownian motion with drift rate 0.1 and variance rate 0.125. Calculate the probability that $X(2)$ is between 0.1 and 0.3.

Solution: $X(2) \sim \hat{\mathrm{N}}(0.2, 0.25)$. Therefore

$$0.1 < X(2) < 0.3 = N\left(\frac{0.3 - 0.2}{0.5}\right) - N\left(\frac{0.1 - 0.2}{0.5}\right) = 0.1585.$$

4.3. The Itô process and Itô's lemma: The Itô process is a generalization of the Wiener process (another name for Brownian motion). You let the coefficients in Eq.(4) above be dependent on t and X.

$$dX = a(X,t)dt + b(X,t)dZ. \tag{6}$$

If a is a constant and $b = \sigma$, you have arithmetic Brownian motion. If $a = 0$ and $b = 1$, you have standard Brownian motion.

Itô's lemma derives a differential equation for a function, $C(X,t)$ of X and t. The crucial point is that in the quadratic Taylor approximation it keeps the term $(dx)^2$ since it is of order dt. This leads to Eq.(16), Chapter 4 of the Study Guide.

Example 4.3.1: Let $\{X(t), t \geq 0\}$ be a Brownian motion with drift coefficient 0.2 and $\sigma = 0.5$. Determine the equation for dC, where $C = tX^2$.

Solution: Use Itô's lemma:

$$\begin{aligned}
\frac{\partial C}{\partial t} &= X^2 = C/t;\ \frac{\partial C}{\partial X} = 2tX = 2\sqrt{tC};\ \frac{\partial^2 C}{\partial X^2} = 2t \\
a &= 0.2,\ b = 0.5 \\
dC &= \left\{(C/t) + 0.4\sqrt{tC} + 0.25t\right\} dt + \sqrt{tC}Z.
\end{aligned}$$

4.4. Geometric Brownian motion. We are often interested in the stochastic process followed by the return of an asset rather than the asset itself. This was made clear in Section 3.4 of the last chapter. In other words, we are interested in $\ln S(t)$, where $S(t)$ is the stock price at time t. $S(t)$ is called a geometric Brownian motion if

$$\frac{dS(t)}{S(t)} = \alpha + \sigma dZ(t). \tag{7}$$

This is an Itô process with $a = \alpha S$ and $b = \sigma S$. By applying Itô's lemma to $\ln S$ one finds that $\ln[S(t)/S(0)]$ is a Brownian motion with (this is a very important point) drift coefficient $\alpha - \sigma^2/2$ and variance rate σ^2. That is,

$$d\ln[X(t)/X(0] = \left(\alpha - \frac{\sigma^2}{2}\right) dt + \sigma dZ. \tag{8}$$

or

$$\ln[X(t)/X(0] = \left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma Z(t). \qquad (9)$$

Example 4.4.1: The current price of a stock is 100. The stock price follows a geometric Brownian motion with drift rate of 10% per year and variance rate of 9% per year. Calculate the probability that two years from now the price of the stock will exceed 200.

Solution: We need $Pr[S(2) > 200] = Pr[S(2)/100 > 2]$ or $Pr[\ln\{S(2)/S(0)\} > \ln 2]$.

$\ln[S(2)/S(0)]$ has a normal distribution with mean $(0.1 - 0.09/2)2 = 0.11$ and variance $(0.09)(2) = 0.18$. Therefore

$$Pr[\ln\{S(2)/S(0)\} > \ln 2] = N\left(\frac{0.6315 - 0.11}{\sqrt{0.18}}\right) = N(1.3745) = 0.9154.$$

Example 4.4.2: (SOA Sample#11) Suppose $S(t)$ follows a geometric Brownian motion. Define $X(t) = \ln[S(t)]$. Which of the following statements are true?

I. $\{X(t), t \geq 0\}$is an arithmetic Brownian motion.

II. $Var[X(t+h) - X(t)] = \sigma^2 h,\ t \geq 0,\ h > 0.$

III. $\lim_{n\to\infty} \sum_{j=1}^{n} [X\left(\frac{jT}{n}\right) - X\left(\frac{(j-1)T}{n}\right)]^2 = \sigma^2 T.$

Solution: All the statements are true.

I follows from Eq.(8) above.

II. is true because X follows a Brownian motion with variance rate σ^2 and therefore $X(t+h) - X(t)$ is normal with variance $\sigma^2 h$.

III is true because Brownian motion is the limiting case of random walk as $n \to \infty$.

Example 4.4.3: (SOA Sample #15:) Let $\{Z(t), t \geq 0\}$ be a standard Brownian motion. You are given:

$$U(t) = 2Z(t) - 2$$

$$
\begin{aligned}
V(t) &= [Z(t)]^2 - t \\
W(t) &= t^2 Z(t) - 2\int_0^t sZ(s)\,ds.
\end{aligned}
$$

Which of the processes defined above have zero drift?

Solution: We need to express the process in the form of Eq.(14) of the Study Guide and check whether or not $a = 0$. The first one is easy.

$$dU = 2dZ(t).$$

So $a = 0$. and there is no drift. For the second one

$$dV = d[Z(t)]^2 - dt.$$

we have to be careful with dZ^2. Use Itô's lemma. Note that Z follows Eq.(14) with $a = 0$ and $b = 1$.

$$\frac{\partial Z^2}{\partial t} = 0; \ \frac{\partial Z^2}{\partial Z} = 2Z; \ \frac{\partial^2 Z^2}{\partial t^2} = 2.$$

Therefore, by Itô's lemma, Eq.(16) with $C = Z^2$. $a = 0$ and $b = 1$,

$$dZ^2 = dt + 2ZdZ$$

and

$$dV = 2ZdZ,$$

and there is no drift.

For the last one, use the fundamental theorem of Calculus and the product rule for differentiation.

$$dW = d[t^2 Z] - 2tZdt = t^2 dZ + 2t2tZdt - 2tZdt = t^2 dZ$$

and again there is no drift.

4.5. Correlated processes: (Sec. 4.5 of the Study Guide.) Perhaps the most important case is when $\rho = 1$. Two processes are driven by the same Brownian motion. In that case the Sharpe ratios have to be equal. You are probably better off by eliminating the Z term and noting that what is left is deterministic and then for no arbitrage the return on the portfolio has to be the risk-free interest. This is what is done in Eqs.(32) and (33) of the Study Guide. In all these non-arbitrage arguments you can manipulate the expressions and eliminate all the terms that involve uncertainty. See Problem #22 at the end of the chapter.

Now please do all the problems at the end of Chapter 4 *except* 11-19 and 23 and then the Quiz.

Quiz 4

1. Let $\{X(t), t \geq 0\}$ be a Brownian motion with zero drift and variance rate (σ^2) 0.1. Calculate $E[X(1)X(2)X(3)]$.

2. Let $\{X(t), t \geq 0\}$ be a Brownian motion with drift coefficient 0.2 and variance rate 0.1. Calculate the probability that the average value of $X(1)$ and $X(3)$ is at most 0.45.

3. $S(t)$ follows
$$d \ln S(t) = 0.12dt + 0.2dZ(t).$$
Given that $S(1) = 100$, calculate $Pr[S(10) < 250]$.

4. $S(t)$ follows:
$$\frac{dS(t)}{S(t)} = 0.12dt + 0.2dZ(t).$$
Given that $S(1) = 100$, calculate $Pr[S(10) < 250]$.

5. The value of currency in country M is currently the same as in country N (i.e., 1 unit in country M can be exchanged for 1 unit in country N). Let $X(t)$ denote the difference between the currency values in country M and N at any point in time (i.e., 1 unit in country M will exchange for $1 + X(t)$ in country N at time t). $X(t)$ is modeled as a Brownian motion process with drift 0 and variance parameter (σ^2) 0.01.

   An investor in country M currently invests 1 in a risk free investment in country N that matures at 1.5 units in the currency of country N in 5 years. After the first year, 1 unit in country M is worth 1.05 in country N.

   Calculate the conditional probability after the first year that when the investment matures and the funds are exchanged back to country M, the investor will receive at least 1.5 in the currency of country M.

6. $\{X(t), t \geq 0\}$ is a geometric Brownian motion with drift coefficient 0.2 and variance rate 0.16. $X(0) = 10$. Let $W = X^2$. Calculate $Pr[W(2) \leq 400]$

7. You are going to gamble at a casino. Your cash position is modeled as a Brownian motion process with a drift rate of -500 per hour and variance rate of $40,000$ per hour. You start at 6 p.m. and end playing at 10 p.m. How much money should you start with so that there is at least a 95% probability that you will not be in debt when you end your gambling session? (Over this time you are allowed to borrow as much as you want.)

8. $S(t)$ follows geometric Brownian motion. Which of the following statements are true?

$$\begin{array}{ll} \text{I.} & Var[\ln S(t+h)|S(t)] = \sigma^2 h, h > 0. \\ \text{II.} & Var\left[\frac{dS(t)}{S(t}|S(t)\right] = \sigma^2\, dt. \\ \text{III.} & Var[S(t+dt)|S(t)] = [S(t)]^2\sigma^2 dt. \end{array}$$

9. (SOA Sample #13:)Consider two non-dividend-paying assets X and Y. There is a single source of uncertainty which is captured by a standard Brownian motion $\{Z(t), t \geq 0\}$. The prices of the assets satisfy the stochastic differential equations

$$\begin{aligned} \frac{dX}{X} &= 0.07\, dt + 0.12 dZ \\ \frac{dY}{Y} &= A dt + 0.085 dZ \end{aligned}$$

The continuously compounded risk-free interest rate is 0.04.

Determine A.

10. The prices of two assets S_1 and S_2 follow geometric Brownian motion.

$$\begin{aligned} dS_1 &= 0.1 S_1 dt + 0.25 S_1 dZ_1 \\ dS_2 &= -0.1 S_2 dt + 0.2 S_2 dZ_2 \end{aligned}$$

A third asset follows

$$dQ(t) = \alpha Q dt + 0.1 Q dZ_1 + 0.1 Q dZ_2.$$

The continuously compounded annual risk-free interest rate is 8%.

Determine α such that there is no arbitrage.

11. Suppose $\{Z_1(t), t \geq 0\}$ and $\{Z_2(t), t \geq 0\}$ are standard Brownian motion processes with a correlation coefficient (per unit time) of ρ determine $dZ_1\, dZ_2$.