Chapter 4

Brownian Motion

4.1. Egbert Sousé Over-served: Egbert Sousé, the Bank Dick portrayed by W.C. Fields, had a penchant for spirituous liquor. One day, unable to stand the thought of facing his loving family after a hard day's work, he enters his favorite tavern, determined to be over-served. And so he is, and by closing time he is totally blotto and is thrown out. He hugs a lamppost and tries to collect his wits to determine which way is home. But to no avail. He starts to walk randomly a step this way now, a step the opposite way later.

Now set up a model for Sousé's movement.

The main goal of constructing a mathematical model is to pass this exam. With that in mind, we will make the following assumptions, without which we will be helpless:

- Despite his condition Sousé traverses a straight line.
- The lamppost is a point, the origin, where Sousé starts. Also Sousé himself is a point! He has no girth as evidenced in the film!
- Sousé takes a step to the right or to the left with equal probability.
- The steps are all identical in **size**.
- The durations between successive steps are all identical.
- The steps he takes are all independent.

This kind of movement has been given an imaginative name. It is called a **Random Walk**. Now we are all set to analyze the motion of Sousé, who, though blotto, walks along a straight line with exactly equal steps at exactly equal times.

What do we mean by, "Set up a model for his movement?" We will take it to mean that we need to describe Sousé's position as a "function" of time t. Call it $X(t)$. I have the word function in quotes because, for each t, $X(t)$ is not a real number. There is uncertainty at each step as to whether he will go to the right or to the left. Therefore we consider $X(t)$, the position at time

t, as a random variable. To set up a model means we have to discover the distribution of $X(t)$.

Suppose Sousé takes n steps over the time interval $(0, t)$. Let the duration for each step be Δt. Thus $\Delta t = t/n$. Denote the change in the position of Sousé over the i-th step by ΔX_i. This is called the increment of $X(t)$ over the duration of the i-th step. We have assumed that the distance travelled over each step is the same, say Δx. Only the direction is random. This means $\Delta X_i = \pm \Delta x$, depending upon whether the step is in one direction or the other. Denote this directional random variable by the symbol Θ_i. Let us set $\Theta_i = 1$ if the step is to the right (in the positive direction) and $\Theta_i = -1$ if the step is to the left (in the negative direction). Then $\Delta X_i = \Delta x \, \Theta_i$. Since Sousé's position at time t, namely $X(t)$, is just the sum of the increments due to all the steps he has taken,

$$X(t) = \sum_{i=1}^{n} \Delta X_i = \sum_{i=1}^{n} \Delta x \ \Theta_i = \Delta x \sum_{i=1}^{n} \Theta_i. \tag{1}$$

Since each step is taken to the right or the left with equal probability (see the third bullet in the list above),

$$Pr\,(\Theta_i = 1) = Pr\,(\Theta_i = -1) = 1/2, \tag{2}$$

It then follows that

$$\begin{aligned} E\,(\Theta_i) &= (1/2)(1) + (1/2)(-1) = 0 && (3) \\ E\left(\Theta_i^2\right) &= 1;\ Var\,(\Theta_i) = 1 && (4) \\ E[X(t)] &= \Delta x \sum_{i=1}^{n} E(\Theta_i) = 0 && (5) \end{aligned}$$

And since the Θ_i's are independent,

$$Var[X(t)] = (\Delta x)^2 \sum_{i=1}^{n} Var(\Theta_i) = n(\Delta x)^2 = \frac{t}{\Delta t}(\Delta x)^2. \tag{6}$$

At this point, for the sake of future development of the theory, let us think of the step size, Δx and the duration Δt as very small - infinitesimally small. Just how small these have to be relative to each other is a very fundamental question, which we will resolve presently.

4.2. The Itô Process: Often when we think of movement, we think of the movement subject to some law such as Newton's law. In fact, in Eq.(11) if σ were zero, that is, if there is no uncertainty, $X(t)$ can be thought of as the position at time t of a mass moving at a constant velocity. $X(0) = 0$ is an initial condition that implies that the mass starts at the origin. μ is the constant velocity. The motion is uniform (unaccelerated) and hence the position $X(t)$ is explicitly given as a linear function of t.

Eq.(11) says that over this steady motion is imposed a random one. This addition makes a very important difference. Whereas in the deterministic motion, with $\sigma = 0$, there is a "velocity" $\mu = X'(t)$, when the Brownian part is added, this is no longer true. $Z(t)$ is not a differentiable function. To see this let us go back to the definition of Brownian motion as a limit of Random Walk. For standard Brownian Motion, $Z(t)$, $\sigma = 1$. So Δz, the step size and Δt approach zero, with $\Delta z/\sqrt{\Delta t}$ fixed as 1, or $\Delta z = \sqrt{\Delta t}$. In this limit, although $Z(t)$ happens to be a continuous function of t (with probability 1), it is not differentiable. The speed, which is the magnitude of the increment divided by the time it takes for the increment, is $\Delta z/\Delta t = 1/\sqrt{\Delta t}$. As Δt approaches 0, this speed becomes infinite! So Sousé is jumping around infinitely fast in a random manner.

We can, however, write Eq.(11) for infinitesimal times and increments as

$$\Delta X(t) = \mu \Delta t + \sigma \Delta Z(t).$$

All you do is replace X, t and Z by dt, dX and dZ. We interpret this as saying that in an infinitesimally small interval, Δt, of time, the increment in $X(t)$ is $\mu \Delta t$ plus σ times the increment of a standard Brownian Motion. Henceforth we will denote the fact that Δt, ΔX and ΔZ are infinitesimally small by replacing them with differentials, dt, dX and dZ. That is, we write:

$$dX(t) = \mu dt + \sigma dZ(t). \tag{13}$$

In order to avoid excursions into rigorous mathematics, I suggest that whenever you see dx, dt, etc., think in terms of small entities such as ΔX, Δt and so on.

In this notation, Eq.(10) becomes[6]

$$(dX)^2 = \sigma^2\, dt. \tag{14}$$

In particular, $(dZ)^2 = dt$.

This means that since $\sqrt{dt}$, is much larger than dt for small dt, **the stochastic term, dominates the process and the drift part is insignificant for small** dt.

We can generalize Eq.(13) by replacing the constants μ and σ with functions that depend upon X and t thus:

$$dX = a(X,t)dt + b(X,t)dZ(t), \tag{15}$$

Again $Z(t)$ follows the standard Brownian motion. A stochastic process satisfying the above equation is called an **Itô process**.

The special case when a and b are constants is Arithmetic Brownian Motion.

If $a(X,t) = \alpha X$ and $b(X,t) = \sigma X,\ \alpha, \sigma$ constants, then the above equation becomes

$$\frac{dX}{X} = \alpha\, dt + \sigma dZ(t). \tag{16}$$

This process is called a **Geometric Brownian Motion**. If X is the stock price, then α is referred to as the expected return on the stock.

A trailer for coming attractions: You cannot replace the left-hand side of Eq.(16) by $d \ln X$ thinking that $d \ln X = (\ln X)' dX$! We shall presently see why.

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[6] Another argument is the following. Consider the square of the increment of a Brownian motion, $(\Delta X)^2$, where $\Delta X = X(t + \Delta t) - X(t)$. Since $\Delta X \sim \mathcal{N}(0, \sigma^2 \Delta t)$, we can set $\Delta X = \sigma\sqrt{\Delta t}\, Y$, where $Y \sim \mathcal{N}(0,1)$.

$$E[(\Delta X)^2] = \sigma^2 \Delta t E(Y^2) = \sigma^2 \Delta t Var(Y) = \sigma^2 \Delta t.$$

On the other hand,

$$Var[(\Delta X)^2] = \sigma^4 (\Delta t)^2 Var(Y^4),$$

which is very small compared to Δt. Therefore in the limit as Δt becomes very small, $Var(\Delta X^2)$ becomes zero and $(dX^2) = E[(dX)^2) = \sigma^2 dt$.

4.3. Polynomial Approximation and Itô's Lemma: Given that $X(t)$ follows an Itô process, we may want to determine the process followed by a function of t and X. For example, given that the price of a stock, $X(t)$, follows an Itô process, we may want to determine the process followed by the price, $C(X,t)$, of an option based on the stock price.

Let us take this in two steps. In the first step we don't look at the process followed by $X(t)$. We concentrate on the given function $C(X,t)$ of the two variables. We treat this as a function in the ordinary sense. For every real number for X and a real number for t, C will give you a unique real number. We first determine the amount dC by which this function changes if we change X by dX and t by dt. I emphasize that the process followed by $X(t)$ does not play any rôle here. You should not even think about the dependence of X on t and Z, which is given by the process equation, until the next step when we will use the process equation for X to put in the appropriate expression for dX in the expression for dC.

To take a trivial example, suppose that $dX = 5dt$ and $C(X,t) = 2X + t$. Then step 1 gives you $dC = 2dX + dt$ and the second step gives you $dC = 2dX + dt = 5dt + dt = 6dt$.

Let us do the first step first. That means we need to express the change in $C(X,t)$ when X and t change by an infinitesimal amount.

In elementary Calculus one writes a Taylor series for a function of two variables, $C(X,t)$. If X changes to $X + dX$ and t to $t + dt$, then the change in C is expressed by

$$\begin{aligned} dC &= C(X+dX, t+dt) - C(X,t) \\ &= C_X(X,t)dX + C_t(X,t)dt \\ &\quad +(1/2)\Big\{C_{XX}(X,t)(dX)^2 + C_{Xt}(X,t)dX\,dt + C_{tX}(X,t)dX\,dt \\ &\quad +C_{tt}(X,t)(dt)^2]\Big\} + \cdots . \qquad (17) \end{aligned}$$

Here the subscripts refer to the partial derivatives. C_x is the derivative of C with respect to X keeping t constant and C_{Xt} is the derivative of C_X with respect to t keeping X constant in the expression for C_X and so on.

Now comes the crucial and exciting part. How far should one go in the Taylor series? The answer is that since dX and dt are infinitesimally small,

Problems

1. Let $Z(t)$ follow the Standard Brownian Motion and $s < t$. Determine $E[Z(s)Z(t)]$.

2. $X(t)$ follows the: Arithmetic Brownian Motion:
$$dX = 0.1dt + 0.2dZ.$$
Calculate $Pr[X(2) > 2|X(1) = 1]$.

3. $\ln X(t)$ follows the Arithmetic Brownian Motion:
$$d\ln X = 0.1dt + 0.2dZ.$$
Calculate $E[X(2)]$.

4. The price of a stock, $S(t)$, follows the geometrical Brownian motion.
$$\frac{dS}{S} = 0.1dt + 0.3dZ.$$
You are given that $S(0) = 100$.

   Calculate $E[S(2)]$.

5. *(Modified) The price of a stock, $S(t)$, follows the geometrical Brownian motion.
$$\frac{dS}{S} = 0.1dt + 0.4dZ.$$
You are given that $S(0) = 100,\ r = 0.08$ and $\delta = 0.10$.

   A forward contract on the stock matures in two years. Let $F(t)$ be the forward price of the stock at time t (in years).

   (a) Write a stochastic differential equation for F.
   (b) Calculate $E[F(1)]$.

6. *A company's cash position (in millions of dollars) follows a generalized Wiener process with a variance rate of 4.0 per quarter. With an initial position of \$2.28 million at the beginning of the year, there is a 14.23% probability that the company will find itself in a negative cash position at the end of the year.

   Calculate the drift rate (assumed constant) per quarter.